Sec 1.6 Substitution

"u sub" for integrals (from Calc)

1) set $u = (x \cdots)$
2) do $du = (\cdots)dx$

2.5) get formula $dx = \frac{du}{(\cdots)}$

3) Replace x, dx with u, du ...
4) Integrate du
5) Replace u with x stuff.

"v sub" for ODEs

1) set $v = (x, y$ stuff $\ldots)$
2) get formula $\frac{dy}{dx} = (x, v, \frac{dv}{dx} \cdots)$
3) replace y, $\frac{dy}{dx}$ with v, $\frac{dv}{dx}$ ...
4) Solve new ODE for v (treating v as dependent on x)
5) Replace $v = (x, y, \cdots)$ in solution
6) Leave implicit solution $F(x,y) = C$,

   or solve for explicit $y = \cdots$

(7? Get particular solution)

Sec 1.6 Substitution

not linear

Ex: Solve ODE $\left\{ \frac{dy}{dx} = (4x+y)^2 \right\}$ $\quad 16x^2 + 8xy + y^2$

not separable

set $v = 4x + y$. $\longleftrightarrow$ $y = v - 3x$

$\frac{dy}{dx} = \frac{dv}{dx} - 4$ $\quad$ implicitly diff

ODE $\quad \frac{dy}{dx} = (4x+y)^2$

sub $\quad \frac{dv}{dx} - 4 = v^2$

$\frac{dv}{dx} = 4 + v^2$ $\quad$ (separable!)

$\int \frac{dv}{4+v^2} = \int dx$

$\tan^{-1}(v/4) = x + C$

$\Longrightarrow \quad v = 4\tan(x+C)$

replace $\quad 4x + y = 4\tan(x+C)$

$y = 4\tan(x+C) - 4x$

For equations

$$\boxed{\frac{dy}{dx} = f(ax+by+c), \text{ use } v = ax+by+C}$$

Ex: Find an implicit sol. for

(not separable, not linear)
$$\frac{dy}{dx} = \frac{2x-y}{x+7y}\cdot\frac{1/x}{1/x}$$

$$\frac{dy}{dx} = \frac{2-(y/x)}{1+7(y/x)} = f(y/x)$$

$\star$ Eqs of form $\frac{dy}{dx} = f(y/x)$ are called <u>homogeneous</u> eqs.

$$\boxed{\text{For homogeneous eqs } \frac{dy}{dx} = f(y/x),\ \text{use } v = y/x}$$

- $v = y/x \quad \longleftrightarrow \quad y = xv$
- $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{2-(y/x)}{1+7(y/x)}$$

$$\cancel{v} + x\frac{dv}{dx} = \frac{2-v}{1+7v} - v = \frac{2-v-v-7v^2}{1+7v}$$

$$x\frac{dv}{dx} = \frac{2-2v-7v^2}{1+7v} \qquad \text{separable}$$

$$\int \frac{(1+7v)\,dv}{2-2v-7v^2} = \int \frac{dx}{x}$$

$u = 2-2v-7v^2$
$du = (-2-14v)\,dv$
$= -2(1+7v)\,dv$

$\rightarrow \quad -\frac{1}{2}\ln|u| = \ln|x| + C$

$\Rightarrow \quad |u| = e^C|x|^{-2}$

$u = Cx^{-2} \qquad (\pm e^C \to C)$

$2-2v-7v^2 = Cx^{-2} \qquad (\text{want } F(x,y) = C)$

$2x^2 - 2x^2\frac{y}{x} - 7x^2\frac{y^2}{x^2} = C$

$\underline{2x^2 - 2xy - 7y^2 = C}$

Equations $\underline{y' + P(x)y = Q(x)y^n}$ are Bernoulli eqs

- $n=0 \Rightarrow y' + Py = Q$ (linear)
- $n=1 \Rightarrow y' + Py = Qy \Rightarrow y' + \underbrace{(P-Q)}_{R}y = 0$ (linear)

(also $y' = -Ry$ separable)

$\boxed{\text{For Bernoulli eqs, use sub } v = y^{1-n}}$ (only needed if $n \neq 0, 1$)

Ex Solve $\{x^2 y' + xy = y^2\}$ assume $x > 0$

$$y' + \frac{1}{x}y = \frac{1}{x^2}y^2 \qquad y' + P \cdot y = Qy^n, \quad n = 2$$

$$v = y^{1-2} = y^{-1} \qquad y = v^{-1}$$

$$\frac{dy}{dx} = -v^{-2}\frac{dv}{dx}$$

$$-v^{-2}\frac{dv}{dx} + \frac{1}{x} \cdot v^{-1} = \frac{1}{x^2}v^{-2}$$

$$\Rightarrow \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2} \qquad v' + P(x)v = Q(x) \text{ linear!}$$

$$P(x) = -1/x, \quad Q(x) = -1/x^2$$

$$\rho = e^{\int P(x)dx} = e^{-\int 1/x\, dx} = e^{-\ln|x|} = |x|^{-1} = x^{-1}$$

$$\rho = x^{-1}$$

$$\rho v' + \rho P v = \rho Q$$

$$x^{-1}v' - x^{-2}v = -x^{-3}$$

$$\int \frac{d}{dx}[x^{-1} \cdot v]\, dx = \int -x^{-3}\, dx$$

$$x^{-1}v = \frac{1}{2}x^{-2} + C$$

$$x^{-1}v = \frac{1}{2}x^{-2} + C$$

$$v = \frac{1}{2}x^{-1} + Cx$$

$$\frac{1}{y} = \frac{1}{2}x^{-1} + Cx$$

$$y = \frac{1}{\frac{1}{2x} + Cx} \cdot \frac{2x}{2x}$$

$$\underline{y = \frac{2x}{1 + Cx^2}}$$